Raising An Axe to the Axioms

An investigation into the definition and foundations of mathematics before the European renaissance

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August 10, 2020

1 Introduction

Mathematics — an art invisibly fundemental to society and yet misunderstood most by its most adept practitioners. As one of the building blocks of civilization, the history of mathematics provides an upper bound to the complexity of any society. Long practiced formally by the educated elite and informally by the laborer, the students of the modern classroom are taught mathematics from the earliest age that adults consider them able. One may conclude by their common sense that the definition of this universal art would be a matter of consensus, as for most that conclusion corresponds with the manner in which they have been taught, a set of computational techniques to memorize and perhaps a set of tools to use to solve problems. One with faith in such a conclusion may be surpised to learn that no such consensus exists [14].

The controversy extends beyond simply the definition. Mathematicians disagree on even basic questions about the nature of the subject and its fundamental essence, such as whether mathematical ideas are constructed or discovered, what activity constitutes mathematics, and the relation of the subject to other areas of study. In this paper, I will give a brief overview of the history of the definition of mathematics and its purpose in various ancient societies, as well as a look at a few often overlooked medieval developments that predate the renaissance.

For brevity, the scope of this paper will be limited mostly to Western history, but a comprehensive treatment of this subject would not rely on such a narrow constraint.

Figure 1: The Ishango Bone [13]

2 Origins

The birth of mathematical practice — like many of the primitive civilizational arts — is cloaked in the mist of prehistory. The oldest known records indicate its invention by reason of necessity. As the human being arose from the animal kingdom and began to plan for the future, hunter-gatherer societies developed a need to measure the passage of time. The earliest record of such an attempt was discovered in Africa in 1960. Known as the Ishango bone, this 20,000-25,000 year-old artifact displays a pattern of groves and notches. While some consider the mammalian bone to be a simply tally stick [13], others suggest that it might be a diagram depicting the phases of the moon [12]. I invite the reader to take a look at figure 1 and make their own judgement.

We can only speculate as to the extent of mathematical thought that is lost to history, either in the form of a lost oral tradition or destroyed document. Some professional and amateur historians may speculate otherwise, but the earliest undisputed documents date back to ancient Sumerian metrology. This usage appears to be fundamentally practical, and this is demonstrated by their used of a sexagesimal number system, later adopted by their Babylonian successors [8]. Similar motivations appear to have inspired the Egyptians, as the bureaucracy needed a technique to manage a complex agricultural system. Far fewer mathematical documents survive from earlier Egyptian society, but the Rhind Mathematical Papyrus, dating back to approximately 1550 BCE, demonstrates — from the perspective of the ancient scribe Ahmose — how to solve eighty-four problems via a practical technique [3]. This is further evidence for the mostly practical and necessity-based nature of early mathematics.

3 Theoretical development

In ancient Greece, we see the emergence of mathematics as a fundamental partner of philosophy. "Tradition" ascribes the following phrase to an engraving above Plato's Academy, "Let no one ignorant of geometry enter". Despite the unfortunate reality that our earliest reliable sources for this inscription appear roughly ten centuries after Plato's death [5] and long after Justinian's order to close the Academy in 529 CE, the spirit of the statement is evident in the work of Plato himself, for he states in the Republic that "studies that demand more toil in the learning and practice than this we shall not discover easily nor find many of them", with reference to the study of geometry [6]. Indeed, it is Plato that established — in the West — the tradition of placing great importance on mathematical education for its own sake. He considered it essential for the development of the mind of the human and its capacity for abstraction. Plato made no significant mathematical discoveries, with the notable exception of the identification of the platonic solids. Rather, his importance to the history of the definition of mathematics is due to the ideological revolution he propagated, for it is his insistence on the relationship between mathematics and philosophy and his emphasis on the importance of free inquiry that forms the very axe which we raise to the axioms of his successors.

The history of Greek deductive reasoning predates and outlives Plato. Aristotle attributes the origin of Greek philosophy to Thales of Miletus [2], and he is the first known individual to use deductive reasoning in a geometric context, as well as the first to be credited with a mathematical discovery [8]. Thales, by his example, paved the way for centuries of Greek pre-science, pulling back the curtain of mythological fantasy with both hands, one fist of natural philosophy, and another of deductive geometry. Centuries of discovery proceeded, from the triangles of Archimedes to the number theory of Diophantus, but the undoubtable crescendo was the Alexandrian Euclid. His *Elements* summarized centuries of Greek progress in Mathematics and served as the most popular and influential mathematics textbook of all time, only being superseded around the late 19th century [8].

We see in *Elements* a great achievement in the history of rigorous reasoning. The textbook formalizes the axiomatic method, a landmark achievement of the human being that — I would argue — deserves similar respect to the scientific method, for the former does for *a priori* objectivity what the latter does for *a posteriori* objectivity. The seed of a giant of rigor is planted by Euclid, and mathematicians ever since have stood on its shoulders. Euclid introduced countless students to methods such as direct proof, reductio ad absurdum, and basic number theory. One can only speculate as to the benefit enjoyed by the human being as a result of this text. Beyond mathematical technique, and very much following in the Platonic tradition of the study thereof as an end in itself, human minds sharpened by the study of Euclid have pierced the veil of numerous fields beyond mathematics. Einstein called the Elements his "holy little geometry book" [1], and by the sheer number of translations, publications, and enthusiasts, as well as a zealous set of followers, this allusion — which in my opinion compares the importance of the *Elements* to that of the Bible — is far from unfounded.

Reverence for mathematics had in fact become a religion of its own for one particular bean-eschewing cult founded by a man with a theorem so eponymous that for many non-mathematicians, the Pythagorean theorem is as well known to them as the fact that the mitochondria is the powerhouse of the cell. As an example of the pop cultural significance of the Pythagoreans, I invite the reader in need of a laugh to feast their eyes upon figure 2.

Pythagoras — incidentally a contemporary of Confucious — was both a mathematician and prophet. Few documents survive from his time period, but historians have reached a consensus on certain points of his life and the activity of his eponymous secret society. His cult is described as similar in form to Orphism but based in mathematics and philosophy. Pythagoras is even credited with the coinage of these very terms, though many of the important discoveries bearing his name, are believed to be discovered by unknown cult-members and held in common [8]. This rare entanglement of religion and mathematics provides an alternative perspective on the ancient definition of the art. In contrast to the Ionian school of Thales, mathematics was not a technique to dispel mythological superstition, rather it was a mystical

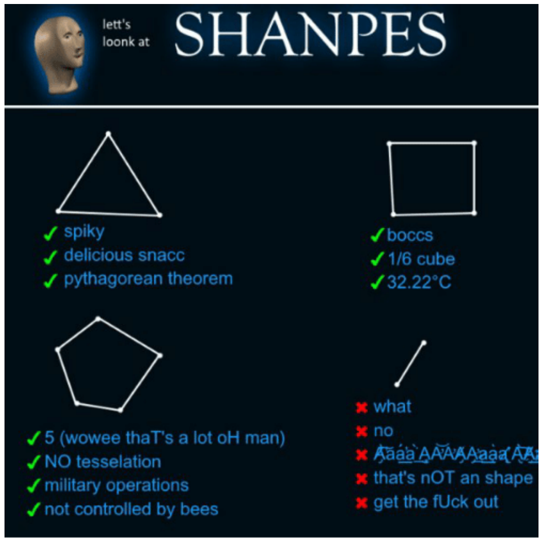

Figure 2: Let's look at Shapnes (Meme/Art) [4]

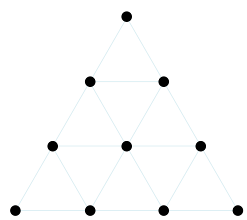

Figure 3: The (holy?) Tetractys [7]

technique in itself. Take for instance the Pythagorean Tetractys, depicted in figure 3. Dantzig relates the following prayer, addressed by the Pythagorean worshipers to the shape of the Tetractys itself with great reverence:

> Bless us, divine number, thou who generated gods and men! O holy, holy Tetractys, thou that containest the root and source of the eternally flowing creation! For the divine number begins with the profound, pure unity until it comes to the holy four; then it begets the mother of all, the all-comprising, all-bounding, the first-born, the never-swerving, the never-tiring holy ten, the keyholder of all [9].

One voice inside of me finds it difficult to relate to this ancient superstition, but this voice gets strangely quiet when I am engrossed in a difficult problem. In those moments of creative flow and arguably elevated consciousness, I can see the Pythagorean perspective. Perhaps not in fourfoldness, but certainly in the sublime nature of mathematical beauty.

Plato and Aristotle, influenced by Pythagorean mysticism, integrated these ideas into their insistence that mathematical understanding is the fundamental building block of higher questions. Indeed, their conception of the universe as being in a harmonic order is arguably derived from Pythagorean beliefs about music and the seemingly perfect ratio between certain notes.

We can describe the ancient definitional controversy with these two broad strokes. On one hand, mathematics is the basis of free scientific inquiry, dispelling superstition and mythology and providing a rational basis for prescience. On the other hand, mathematics is itself a spiritual tool, providing insight into mystical metaphysical realms and a glimpse beyond the physical world. Viewed from a certain anggle, this dichotomy has a correspondence with one of the fundamental definitional dichotomies of modern math, the question of whether mathematical concepts are discovered or invented. If math is invented, then it is one of the greatest constructions of the human being, providing a basis for nearly all other constructions and creations, demonstrating the power of human creativity and the ability of the human to rise beyond the animal constraints of the mind so imposed and hardwired by the primordial evolutionary conditions. If math is discovered, then it is one of the greatest mysteries the human has uncovered, a secret truth so fundamental to the universe that the study of the art corresponds with later *a posteriori* scientific discovery long before the necessary observations and deductions are made.

Mathematics virtually disappears from Western Europe, along with literacy and scholarship, following the decline and fall of the Roman empire. Though the art thrived in many more sophisticated parts of the world throughout the period following the collapse, I would argue that mathematics reappears in Western Europe far earlier than most give credit. The traditional history of Western European civilization, twisted by the early Italian renaisaince rejection of catholic dogma in all its forms, may have thrown out important chunks of baby with its dank, superstitious bathwater.

Euclid reappears in Western Europe in the twelfth century thanks to Adelard of Bath, who visited Islamic Spain and made the translation from Arabic to Latin, introducing the lost axiomatic method of the ancients to the monastic orders of Europe, who — though highly superstitious and constrained by the Catholic Church — began to revive the Academic tradition of Plato and Aristotle during the Scholastic movement of that and the following century [11]. Russell — a great mathematician himself — criticizes the Scholastics for their obsession with dialectic, even in contradiction to observational evidence. Certainly this is unscientific and this manner of thinking is a dead end for much of natural philosophy, but their emphasis on "verbal distinctions and subtleties" and intense focus on derivations from an infallible scripture appear as a pre-mathematics in parallel to the reappearance of Euclid in the west. Abélard, a medieval French Scholastic, serves as a prime example of this pre-renaissance renaissance in axiomatic rigor. Despite a striking and unfortunate disdain for empiricism, his book "Yes and No", composed in the early twelfth century, attempts to argue for numerous theses, not so much in order to reach a conclusion as to demonstrate the beauty and importance of argument itself. He was such a fanatic of Logic that he considered it to be the most Christian of the sciences, and compared the word to the Christian conception of Jesus as the Greek Logos, the terminology used in Saint John's gospel [11].

This is a definitional novelty in the history of mathematics. Despite the vast intellectual oppression of Catholic dogma, the imposed constraints of scripture served as state-mandated axioms, and rigorous, narrow, systematic *a priori* reasoning was applied to this non-numerical and non-geometric system of thought, introducing a new vector of abstraction in the very axioms of the system itself. To Euclid, an axiom was simply a common notion that anyone could — and should — accept as true before actual argument, and to the Scholastics, scripture and dogma served as axioms themselves, as they had little ideological choice. Even in contrast to the Pythagoreans this was novel, as it was not number and form that served as the metaphysical basis, rather it was dialectical study of a non-numerical axiomatic system, applying the methods of mathematics to a previously non-mathematical context. This ideological novelty appears far before the more famous usage of non-numerical analysis of our time, that of the analysis of algorithm, and perhaps this early medieval study of precision in verbal subtlety and systematization of non-numerical argument contributed to later systems of thought as vast as formal language theory and alternative axiomatic systems. If so, this is a relatively unknown revolution in mathematical thinking, as the death of certainty in scripture in later European academia drained these arguments of their ideological weight while preserving their rigor as isolated abstractions, consistent only within the context of their own argumentative reasoning but disconnected to the greater universe. This is the modern notion of an axiomatic system, and perhaps the medieval Scholastics deserve more credit than they are given.

4 Conclusion

Unfortunaltely, we set down our axe before the forest has fully grown. Mathematical complexity exploded parallel to the scientific revolution, and development of the axiomatic method and rigorous foundations for mathematics co-developed with scientific rigor. The two great areas of endeavor appear to be fundamentally entwined, as Wigner argues in his 1960 article on the "Unreasonable Effectiveness of Mathematics in the Natural Sciences". Mathematicians often find pure mathematical results that are only later found to be applicable to some aspect of the natural world [15].

At the same time, mathematics has its controversies. Most of what we call modern mathematics is founded — sometimes explicitly but often implicitly — in the Zermelo-Fraenkel set theory axioms, generally including an additional axiom known as the axiom of choice. As demonstrated by the Scholastics, the axiomatic method can be applied to areas of study beyond the numeric and geometric, and as proponents of alternative foundations such as Homotopy Type Theory argue, modern mathematics can and perhaps should move past the default choice of a set-theoretic foundation [10]. Unlike the scripture of those tireless medieval monk, modern mathematics was not delivered in complete and perfect form to a prophet on a mountain to be forever engraved in stone. Rather, the story of mathematics continues to be written and debated, with control of the art resting in the hands of no single individual. Mathematics is the collective complexity of the human being and like the human, its greatest quality is its capacity to change.

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